PRACTICAL CONSIDERATIONS IN HUMAN-INDUCED VIBRATION

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Introduction

This document provides a review of fundamental concepts in structural dynamics and some applications in human-induced vibration analysis and mitigation of structures.

Modal Analysis of MDOF Structures

Consider a general *n*–DOF system defined by

$$\mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) = \mathbf{F}(t) \tag{1}$$

Where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the $n \times n$ mass, damping and stiffness matrices, $\mathbf{F}(t)$ is the $n \times 1$ external excitation, and $\mathbf{x}(t)$ is the $n \times 1$ displacement vector. Herein, $\mathbf{M}$ can be a diagonal matrix with proper choices of the reference frames. In practical applications, $\mathbf{M}$ and $\mathbf{K}$ matrices are computed using finite element (FE) procedures, $\mathbf{C}$ is predefined. It can be shown that the structural response, $\mathbf{x}(t)$ can be written in terms of any modal matrix, $\mathbf{\Phi}$ and the normal coordinates, $\mathbf{q}(t)$ as

$$\mathbf{x}(t) = \mathbf{\Phi}\mathbf{q}(t) \quad \text{and} \quad \dot{\mathbf{x}}(t) = \mathbf{\Phi}\dot{\mathbf{q}}(t) \quad \text{and} \quad \ddot{\mathbf{x}}(t) = \mathbf{\Phi}\ddot{\mathbf{q}}(t) \tag{2}$$

where

$$\mathbf{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_n \end{bmatrix} \quad \text{and} \quad \boldsymbol{\phi}_i = \begin{bmatrix} \phi_i^1 \\ \phi_i^2 \\ \cdots \\ \phi_i^n \end{bmatrix} \tag{3}$$

where $\boldsymbol{\phi}_i$ is the i^{th} mode shape. Substituting (2) into (1) and pre-multiplying by $\mathbf{\Phi}^{\mathrm{T}}$, another form of the equation of motion is obtained as

$$\mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}\ddot{\mathbf{q}}(t) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}\dot{\mathbf{q}}(t) + \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi}\mathbf{q}(t) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F}(t) \tag{4}$$

It is convenient to define new structural matrices as

$$\overline{\mathbf{M}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{M}\mathbf{\Phi}, \quad \overline{\mathbf{C}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{C}\mathbf{\Phi}, \quad \overline{\mathbf{K}} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{K}\mathbf{\Phi} \quad \text{and} \quad \overline{\mathbf{F}}(t) = \mathbf{\Phi}^{\mathrm{T}}\mathbf{F}(t) \tag{5}$$

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where $\bar{\mathbf{M}}$, $\bar{\mathbf{C}}$ and $\bar{\mathbf{K}}$ are called *generalized mass, damping* and *stiffness matrices*, respectively, and $\bar{\mathbf{F}}$ is the *generalized excitation*. Due to orthogonality of the mode shapes, these new structural matrices are diagonal, *i.e*.

$$\bar{\mathbf{M}} = \begin{bmatrix} \bar{m}_1 & & & \\ & \bar{m}_2 & & \\ & & \cdots & \\ & & & \bar{m}_n \end{bmatrix}, \quad \bar{\mathbf{C}} = \begin{bmatrix} \bar{c}_1 & & & \\ & \bar{c}_2 & & \\ & & \cdots & \\ & & & \bar{c}_n \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{K}} = \begin{bmatrix} \bar{k}_1 & & & \\ & \bar{k}_2 & & \\ & & \cdots & \\ & & & \bar{k}_n \end{bmatrix} \tag{6}$$

It should be noted that since the damping matrix $\mathbf{C}$ is predefined in such a way that the generalized damping matrix $\bar{\mathbf{C}}$ has a diagonal form for ease of mathematical computations. This type of damping is know as classical damping. Generalized matrices do not have much practical meaning in this form. However, if the modal matrix, $\mathbf{\Phi}$ is selected in a way that mode shapes are mass orthonormalized, the generalized structural matrices take a more elegant and useful form as

$$\bar{\mathbf{M}} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \cdots & \\ & & & 1 \end{bmatrix} = \mathbf{I}, \quad \bar{\mathbf{C}} = \begin{bmatrix} 2\zeta_1\lambda_1 & & & \\ & 2\zeta_2\lambda_2 & & \\ & & \cdots & \\ & & & 2\zeta_n\lambda_n \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{K}} = \begin{bmatrix} \lambda_1^2 & & & \\ & \lambda_2^2 & & \\ & & \cdots & \\ & & & \lambda_n^2 \end{bmatrix} \tag{7}$$

where ζ_i and λ_i are the modal damping ratio and the frequency of the i^{th} mode. Therefore, while a mass orthonormalized modal matrix yields an identity generalized mass matrix, a randomly selected modal matrix yields merely a diagonal generalized mass matrix.

It is very important to note that modal damping ratios and frequencies are independent of the selected modal matrix. Therefore, to characterize the behavior of a mode, one only needs the modal damping ratio and frequency; generalized mass, $\bar{m}$ is not required. To characterize the behavior of the overall MDOF system, one needs the modal matrix that is used in the eigenvalue analysis of the equation of motion in addition to the modal damping ratios and frequencies.

Modal Time-History Analysis of MDOF Structures

There are several numerical methods to solve the matrix differential equation (1). One of this method is to solve (1) directly with explicit or implicit numerical schemes such as Newmark's β method and Runge-Kutta method, which can be called *time-history analysis*. Another method is to solve the modal equations that results from the modal analysis, and then to combine the modal responses to find the final structural responses. This method can be called *modal time-history analysis*. Modal time-history analysis is considerably time-effcient and more frequently used for higher degree-of-freedom systems compared to time-history analysis. However, a typical modal time-history analysis always require a diagonal damping matrix, while regular time-history analysis can solve systems with any type of damping matrix. In the following, a review of the modal time-history analysis is given.

The modal equations obtained from equation (4) can be written as

$$\begin{aligned} \text{1st mode:} \quad & \bar{m}_1\ddot{q}_1(t) + \bar{c}_1\dot{q}_1(t) + \bar{k}_1 q_1(t) = \bar{F}_1(t) \\ \text{2nd mode:} \quad & \bar{m}_2\ddot{q}_2(t) + \bar{c}_2\dot{q}_2(t) + \bar{k}_2 q_2(t) = \bar{F}_2(t) \\ & \dots \\ \text{n-th mode:} \quad & \bar{m}_n\ddot{q}_n(t) + \bar{c}_n\dot{q}_n(t) + \bar{k}_n q_n(t) = \bar{F}_n(t) \end{aligned} \tag{8}$$

Each of these equations can be solved using regular time-history method, and final structural responses can be found using equations (2).

Effective Mass

This section provides a general form of an effective mass of a lumped-mass MDOF system that vibrates in a predefined shape. Note that this shape is arbitrary and does not need to be a modal shape of the system.

Consider the lumped-mass, *n*–DOF system shown in Fig. 1, and assume that it vibrates arbitrarily as shown in Fig. 2 such that the deformation of the structure will follow a fixed shape, *i.e*.

$$x_i(t) = (\text{constant})x_j(t)\,, \quad \text{for all } i \text{ and } j \text{ at time } t \tag{9}$$

where $x_i(t)$ is the displacement of mass m_i at time t. The displacement of any joint can be written in terms of a unique time-function, $q(t)$ and a constant, a_i as $x_i = a_i q(t)$. Also, $x_j = a_j q(t)$. In this case, the overall displacement of the structure can be written in a vector form as

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_i(t) \\ \dots \\ x_n(t) \end{bmatrix} \quad \text{or} \quad \mathbf{x} = \begin{bmatrix} a_1 q(t) \\ a_2 q(t) \\ \dots \\ a_i q(t) \\ \dots \\ a_n q(t) \end{bmatrix} = \mathbf{a}q(t) \quad \text{where} \quad \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ a_i \\ \dots \\ a_n \end{bmatrix} \tag{10}$$

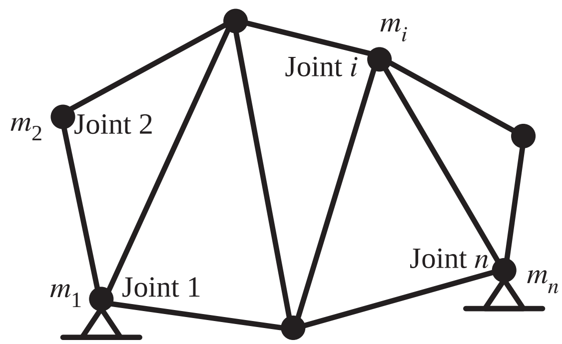

FIG. 1 A general lumped-mass n–DOF system

Now consider the following problem: Find a realization of the n–DOF system given in Fig. 1 such that the equivalent system is characterized with the vibration of only one DOF (instead of n DOFs) and an effective mass is attached to that joint. Such a realization is shown in Fig. 3. To find the effective mass, m^i_{eff}, the kinetic energy of the original system is equated to the kinetic energy of the equivalent system at time t. The kinetic energy of the original system is given by

$$\begin{aligned} E_{\text{original}} &= \frac{1}{2}m_1(\dot{x}_1(t))^2 + \frac{1}{2}m_2(\dot{x}_2(t))^2 + \cdots + \frac{1}{2}m_n(\dot{x}_n(t))^2 \\ &= \frac{1}{2}m_1(a_1\dot{q}(t))^2 + \frac{1}{2}m_2(a_2\dot{q}(t))^2 + \cdots + \frac{1}{2}m_n(a_n\dot{q}(t))^2 \\ &= \frac{1}{2}(m_1a_1^2 + m_2a_2^2 + \cdots + m_na_n^2)(\dot{q}(t))^2 \\ &= \frac{1}{2}(\dot{q}(t))^2\sum_{k=1}^{n} m_ka_k^2 \end{aligned} \tag{11}$$

The kinetic energy of the equivalent system is

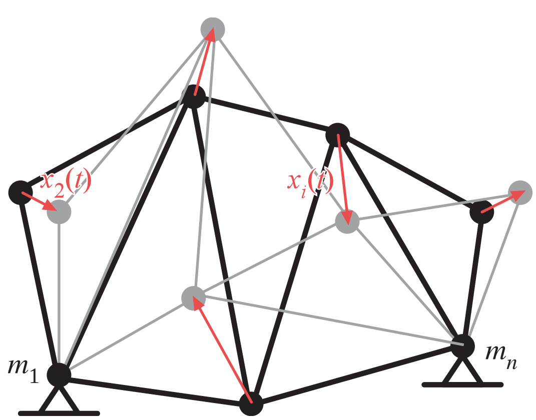

FIG. 2 Arbitrary vibration

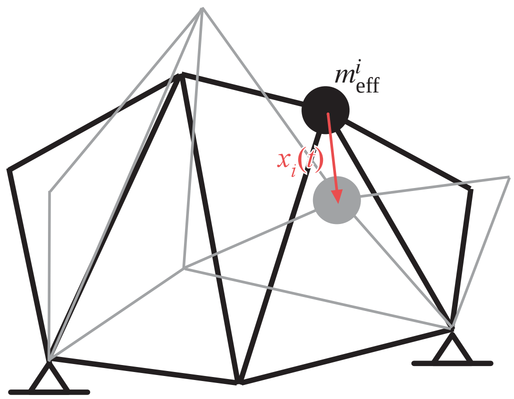

FIG. 3 An equivalent system with single displacement

$$\begin{aligned} E_{\text{equiv}} &= \frac{1}{2} m^i_{\text{eff}} (x_i(t))^2 \\ &= \frac{1}{2} m^i_{\text{eff}} a_i^2 (\dot{q}(t))^2 \end{aligned} \tag{12}$$

Equating the kinetic energies yields

$$m^i_{\text{eff}} = \frac{\sum_{k=1}^{n} m_k a_k^2}{a_i^2} \tag{13}$$

Equation (13) can be used to find the effective masses of various vibration shapes.

Effective Mass for a Mode Shape

In this section, first, a review of mode shapes and generalized masses is given. Then, an effective mass for a given mode shape is found.

In the following, effective masses for two cases are derived: (a) a randomly selected modal matrix and (b) a mass orthonormalized modal matrix.

Consider the problem defined in the previous section. Let an *n*–DOF structure vibrate in its j^{th} mode shape instead of an arbitrary predefined shape, *i.e.*, $\mathbf{a} = \phi_j$ or $a_k = \phi_j^k$ for all *k*. Please note that the selected mode matrix $\mathbf{\Phi}$ is randomly selected and is not necessarily a mass orthonormalized mode matrix. It can be shown with some matrix algebra that the generalized mass of the j^{th} mode in (6) is indeed given by

$$\bar{m}_j = \sum_{k=1}^{n} m_k (\phi_j^k)^2 \tag{14}$$

Therefore, the effective mass for the j^{th} mode and i^{th} displacement is given by

$$m_{j,\text{ eff}}^i = \frac{\bar{m}_j}{(\phi_j^i)^2} \tag{15}$$

If the mode shapes are selected such that the modal matrix is mass orthonormalized as given in (7), then

$$\bar{m}_j = 1 \tag{16}$$

and the effective mass for this special case is given by

$$m_{j,\text{ eff}}^i = \frac{1}{(\phi_j^i)^2} \tag{17}$$

This completes the derivation of the effective masses. In the following section, effective masses are derived thorugh the equation of motion of the equivalent system.

Response to an Excitation

In this section, a generalized form of the effective mass computation is given when an n–DOF, lumped-mass structure vibrates in a predefined arbitrary shape and excited by an arbitrary force at a given DOF. Then, a special case is considered, where the vibration shape is taken as one of the a mode shapes of the structure instead of an arbitrary shape. Also given for this special case is the peak responses when the excitation is a sinusoidal function.

Let the n–DOF structure shown in Fig. 1 vibrate in an arbitrary shape defined by equations (10) with an external force $F(t)$ applied at j^{th} DOF as shown in Fig. 4. Note that the force $F(t)$ is excited exactly in the direction of x_j. The governing equation of motion of this vibration can be found easily by applying virtual energy principles as

$$\frac{1}{a_j}\left(\sum_{k=1}^{n} m_k a_k^2\right)\ddot{q} + \frac{1}{a_j}\mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}\dot{q} + \frac{1}{a_j}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}q = F(t) \tag{18}$$

which can be written as

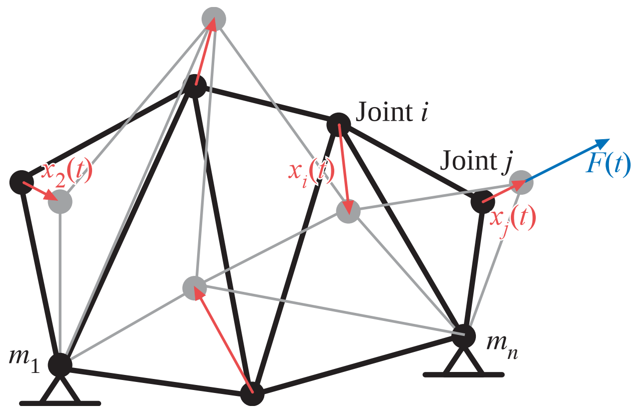

FIG. 4 Arbitrary vibration with an excitation

$$\left(\sum_{k=1}^{n} m_k a_k^2\right)\ddot{q} + \mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}\dot{q} + \mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}q = a_j F(t) \qquad (19)$$

where $\mathbf{C}$ and $\mathbf{K}$ are the damping and stiffness matrices of the n–DOF system, respectively.

Now, consider a system that is equivalent to the n–DOF system, which is given in Fig. 4, such that an effective mass is attached to the i^{th} DOF, and an external force is applied at the j^{th} DOF. This equaivalent system is shown in Fig. 5. The governing equation of motion of the equivalent system can also be derived using virtual energy principles as

$$\frac{a_i^2}{a_j} m_{\mathrm{eff}}^i \ddot{q} + \frac{1}{a_j}\mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}\dot{q} + \frac{1}{a_j}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}q = F(t) \qquad (20)$$

which can be written as

$$a_i^2 m_{\mathrm{eff}}^i \ddot{q} + \mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}\dot{q} + \mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}q = a_j F(t) \qquad (21)$$

Therefore, the effective mass at the i^{th} DOF for a forced vibration at the j^{th} DOF is given by

$$m_{\mathrm{eff}}^i = \frac{\sum_{k=1}^{n} m_k a_k^2}{a_i^2} \qquad (22)$$

The response of the overall structure can be found by two methods. In the first method, the equation (21) is solved for $q(t)$, or equivalently, the equation

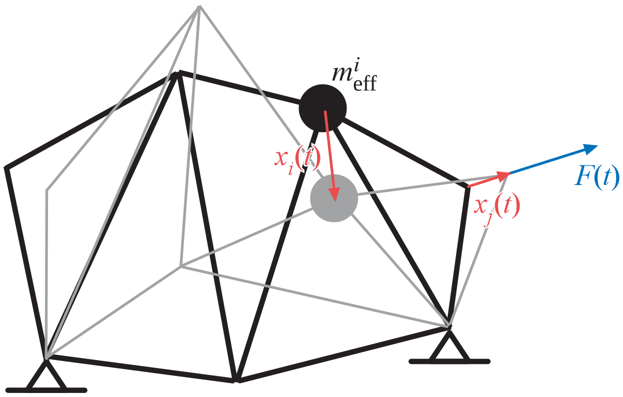

FIG. 5 Equivalent SDOF system for the forced vibration

$$m^i_{\text{eff}}\ddot{q} + \frac{1}{a_i^2}\mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}\dot{q} + \frac{1}{a_i^2}\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}q = \frac{a_j}{a_i^2}F(t) \tag{23}$$

is solved for q. Then, the structural responses are given by $\mathbf{x} = \mathbf{a}q$, *i.e.* $x_l = a_l q$ for any DOF l.

In the second method, the governing equation of motion of the equivalent SDOF system is written as

$$m^i_{\text{eff}}\ddot{x}_i + \frac{\mathbf{a}^{\mathrm{T}}\mathbf{C}\mathbf{a}}{a_i}\dot{x}_i + \frac{\mathbf{a}^{\mathrm{T}}\mathbf{K}\mathbf{a}}{a_i}x_i = \frac{a_j}{a_i}F(t) \tag{24}$$

and solved for x_i and the response of structure is then found as

$$\mathbf{x} = \frac{1}{a_i}\mathbf{a}x_i, \quad i.e. \quad x_l = \frac{a_l}{a_i}x_i \tag{25}$$

Next, the equivalent SDOF is found for a special case where the arbitrary shape is set to one of the mode shapes of the structure. Note that the mode shape is not necessarily mass or displacement normalized. Let the mode shape in consideration be the m^{th} mode, *i.e.* $\mathbf{a} = \boldsymbol{\phi}_m$. The equation of motion of the n–DOF structure becomes

$$\left(\sum_{k=1}^{n} m_k(\phi_m^k)^2\right)\ddot{q} + \boldsymbol{\phi}_m^{\mathrm{T}}\mathbf{C}\boldsymbol{\phi}_m\dot{q} + \boldsymbol{\phi}_m^{\mathrm{T}}\mathbf{K}\boldsymbol{\phi}_m q = \phi_m^j F(t) \tag{26}$$

which can be simplified to

$$\overline{m}_m\ddot{q} + 2\overline{m}_m\zeta_m\omega_m\dot{q} + \overline{m}_m\omega_m^2 q = \phi_m^j F(t) \tag{27}$$

where $\overline{m}_m$, ζ_m are ω_m given by (7). Similarly, the equation of motion of the equivalent SDOF system becomes

$$(\phi_m^i)^2 m_{m,\,\text{eff}}^i \ddot{q} + 2\overline{m}_m \zeta_m \omega_m \dot{q} + \overline{m}_m \omega_m^2 q = \phi_m^j F(t) \tag{28}$$

Comparing equations (27) and (28), the effective mass that is attached to the i^{th} DOF of the structure, which is vibrating in the m^{th} mode is found to be

$$m_{m,\,\text{eff}}^i = \frac{\overline{m}_m}{(\phi_m^i)^2} \tag{29}$$

As explained before, the structural responses can be found in two ways. In the first method, the equation of motion of the equivalent SDOF is written in terms of q as follows:

$$m_{m,\,\text{eff}}^i \ddot{q} + 2m_{m,\,\text{eff}} \zeta_m \omega_m \dot{q} + m_{m,\,\text{eff}} \omega_m^2 q = \frac{\phi_m^j}{(\phi_m^i)^2} F(t) \tag{30}$$

and the structural responses are found by solving (30) for q and are given by $\mathbf{x} = \boldsymbol{\phi}_m q$, *i.e.* $x_l = \phi_m^l q$ for any DOF *l*.

In the second method, equation (28) is rewritten in terms of x_i as

$$m_{m,\,\text{eff}}^i \ddot{x}_i + 2m_{m,\,\text{eff}} \zeta_m \omega_m \dot{x}_i + m_{m,\,\text{eff}} \omega_m^2 x_i = \frac{\phi_m^j}{\phi_m^i} F(t) \tag{31}$$

and the structural responses are found by solving (31) for x_i and are given by

$$\mathbf{x} = \frac{1}{\phi_m^i} \boldsymbol{\phi}_m x_i, \quad i.e. \quad x_l = \frac{\phi_m^l}{\phi_m^i} x_i \tag{32}$$

It is easy to note that the two methods presented above are identical except that former one uses normalized modal responses while the latter uses the structural responses in the equation of motion, which results different forcing function coefficients, and therefore same structural responses.

Now, consider a special case where the excitation is a sinusoidal function and is defined by

$$F(t) = P \sin(\omega_m t) \tag{33}$$

where ω_m is the natural frequency of the m^{th} mode. The structural responses can conveniently found by the equations defined above. In addition, peak structural responses can be computed using the same equations. As an example, consider equation (30) and the peak structural acceleration. The peak structural acceleration corresponds to the peak of $\ddot{q}$, which can be found as

$$\ddot{q}^{\text{peak}} = \left(\frac{\phi_m^j}{(\phi_m^i)^2}\right)\frac{P}{2m_{m,\text{ eff}}^i\zeta_m} \tag{34}$$

which is simplified using (29) to

$$\ddot{q}^{\text{peak}} = \phi_m^j\frac{P}{2\bar{m}_\text{m}\zeta_m} \tag{35}$$

and the peak structural responses are given by

$$\ddot{\mathbf{x}}^{\text{peak}} = \boldsymbol{\phi}_m\ddot{q}^{\text{peak}}, \quad i.e. \quad \ddot{x}_l^{\text{peak}} = \phi_m^l\ddot{q}^{\text{peak}} = \phi_m^l\phi_m^j\frac{P}{2\bar{m}_\text{m}\zeta_m}. \tag{36}$$

Also consider the equation (31). The peak structural acceleration $\ddot{x}_i^{\text{peak}}$ is given by

$$\ddot{x}_i^{\text{peak}} = \left(\frac{\phi_m^j}{\phi_m^i}\right)\frac{P}{2m_{m,\text{ eff}}^i\zeta_m} \tag{37}$$

which simplifies to

$$\ddot{x}_i^{\text{peak}} = \phi_m^j\phi_m^i\frac{P}{2\bar{m}_\text{m}\zeta_m} \tag{38}$$

and the peak structural accelerations are given by

$$\ddot{\mathbf{x}}^{\text{peak}} = \frac{1}{\phi_m^i}\boldsymbol{\phi}_m x_i^{\text{peak}}, \quad i.e. \quad \ddot{x}_l^{\text{peak}} = \frac{\phi_m^l}{\phi_m^i}\ddot{x}_i^{\text{peak}} = \phi_m^l\phi_m^j\frac{P}{2\bar{m}_\text{m}\zeta_m} \tag{39}$$

which is identical to (36). These results show that peak structural responses are independent of the effective mass in the case of a special vibration, where the structure deformation is defined by one of its modes.

Also note that if the modal shapes, $\boldsymbol{\phi}$ are selected such that they are mass orthonormalized, then in the above equations $\bar{m}_\text{m} = 1$.

One important observation is that the only required information to compute peak accelerations is the mode shapes and corresponding generalized masses. Different normalization methods do not provide any significant convenience.

Summary

This document provides the computation of effective masses for a general n-DOF, lumped-mass system when it vibrates in a predefined shape. Special cases are considered when the predefined shape is selected as one of the mode shapes of the structure and an excitation is applied at a point in the direction of the corresponding DOF. Above derivations are straightforward for continuous systems. The results are summarized in Table 1.

TABLE 1 Effective masses, generalized masses and peak accelerations for different vibration shapes

Vibration Characteristics	Vibration Vector	Generalized Mass	Effective Mass	Peak Acceleration[a]
Arbitrary	$\mathbf{a}$	$\sum_{k=1}^{n} m_k a_k^2$	$m_{\text{eff}}^i = \frac{\sum_{k=1}^{n} m_k a_k^2}{a_i^2}$	Solve (23) or (24)
m^{th} mode (not normalized)	ϕ_m	$\bar{m}_m = \sum_{k=1}^{n} m_k (\phi_m^k)^2$	$m_{m,\text{ eff}}^i = \frac{\bar{m}_m}{(\phi_m^i)^2}$	$\ddot{x}_l^{\text{peak}} = \phi_m^l \phi_m^j \frac{P}{2\bar{m}_{\text{m}}\zeta_m}$
m^{th} mode (displacement normalized)[b]	ϕ_m	$\bar{m}_m = \sum_{k=1}^{n} m_k (\phi_m^k)^2$	$m_{m,\text{ eff}}^i = \frac{\bar{m}_m}{(\phi_m^i)^2}$	$\ddot{x}_l^{\text{peak}} = \phi_m^l \phi_m^j \frac{P}{2\bar{m}_{\text{m}}\zeta_m}$
m^{th} mode (mass normalized)[c]	ϕ_m	$\bar{m}_m = 1$	$m_{m,\text{ eff}}^i = \frac{1}{(\phi_m^i)^2}$	$\ddot{x}_l^{\text{peak}} = \phi_m^l \phi_m^j \frac{P}{2\zeta_m}$

a. For excitation $F(t) = P\sin(\omega_m t)$ applied at DOF j
b. This is the standard procedure in GSA
c. This is the standard procedure in SAP2000 and alternate procedure in GSA